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The supersymmetry of the (1 + 1)-dimensional oscillators in general relativity

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Abstract. The quantum modes of a new family of relativistic oscillators are studied by using the supersymmetry and shape invariance in a version suitable for (1 + 1)-dimensional relativistic systems. In this way one obtains the Rodrigues formulae of the normalized energy eigenfunctions of the discrete spectra and the corresponding raising and lowering operators.

1. Introduction

In general relativity, the geometric models play the role of kinetics, helping us to understand the characteristics of the classical or quantum free motion on a given background. One of the simplest geometric models in (1 + 1) dimensions is that of the quantum relativistic oscillator (RO) defined as a free massive scalar particle on the anti-de Sitter static background [1–3]. Recently, we have generalized this model to a family of quantum models of ROs, whose metrics are one-parameter deformations (i.e. conformal transformations) of the anti-de Sitter or de Sitter ones [4]. In the case of the deformed anti-de Sitter metrics we have considered that the backgrounds are, in fact, the universal covering spacetimes of the globally hyperbolic original ones such that their time coordinates cover the whole real axis [5]. Thus we enssure the geometric compatibility of all the models given by our one-parameter family of metrics. As shown in [6], the deformed anti-de Sitter metrics lead to relativistic equivalents of the usual non-relativistic Pöschl–Teller (PT) problems, while the deformed de Sitter metrics generate relativistic Rosen–Morse (RM) problems [7]. A remarkable property of these ROs is that all of them have the same non-relativistic limit in the sense of special relativity, namely just the usual non-relativistic harmonic oscillator (NRHO) [4].

The Klein–Gordon equation of these models is analytically solvable in the same manner as the Schrödinger equation of the mentioned well studied non-relativistic problems. This allows one to study the RO by using the successful methods of supersymmetry and shape invariance [8] with the minimal changes requested by the specific form of the Klein–Gordon equation [9]. In this way one can derive the normalized energy eigenfunctions of the discrete energy spectrum and the form of the shift operators of the energy basis that are involved in the structure of the dynamical algebras [10].

Here we would like to present a systematic study of our family of ROs based on the supersymmetry and shape invariance of the relativistic potentials, pointing out the main specific features of the PT and RM relativistic problems. We believe that this first example of a family of metrics generating analytically solvable quantum problems could be of interest for further investigations concerning the supersymmetry of other solvable relativistic quantum models in (3 + 1) dimensions [5, 11] or more [12].

We start in section 2 with a short review of the relativistic scalar quantum mechanics in (1 + 1) dimensions. In section 3 we present the relativistic PT and RM oscillators giving their energy spectra and the energy eigenfunctions up to normalization factors. The relativistic supersymmetry and the shape invariance of the relativistic PT and RM potentials are used in the following section for deriving the definitive form of the normalized energy eigenfunctions of the discrete energy spectra. Section 5 is devoted to the properties of the shift operators of the energy bases of our RO. Therein we recover the known shift operators of the PT models [13] and we write down those of the RM models. We use natural units with $\hbar = c = 1$.

2. Preliminaries

It is well known that one-particle relativistic quantum mechanics cannot be constructed as an independent consistent theory because of some difficulties related to the probabilistic interpretation of the relativistic wavefunctions. For this reason, we consider the relativistic quantum mechanics as the one-particle restriction of the theory of the scalar quantum field on curved backgrounds [14, 15].

Let us start with a (1 + 1)-dimensional background with a static local chart (i.e. natural frame) of coordinates $(u^0, u^1) \equiv (t, u)$ where the metric tensor defined on the space domain D_u is $g_{\mu\nu}(u)$, $\mu, \nu = 0, 1$. The one-particle quantum modes of the scalar field ϕ of the mass m, minimally coupled with the gravitational field, are given by the Klein–Gordon equation

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0 \qquad g = |\det(g_{\mu\nu})|. \tag{1}$$

Since in the static charts the energy, E, is conserved, this equation has a set of fundamental solutions (of positive and negative frequencies),

$$\phi_E^{(+)}(t, u) = \frac{1}{\sqrt{2E}}\mathrm{e}^{-\mathrm{i}Et}U_E(u) \qquad \phi^{(-)} = (\phi^{(+)})^* \tag{2}$$

which depend on the static energy eigenfunctions U_E. These may be orthonormal (in an usual or generalized sense) with respect to the relativistic scalar product [14]

$$\langle U, U'\rangle = \int_{D_u} \mathrm{d}u\, \mu(u)U(u)^*U'(u) \tag{3}$$

defined by the relativistic weight function of the scalar field, $\mu = \sqrt{g}\, g^{00}$.

It is known that in (1 + 1) dimensions any static background has *special* natural frames, where the metric is the conformal transformation of the flat one. Starting with any natural frame (t, u), the space coordinate of the special frame (t, x) reads

$$x = \int \mathrm{d}u\, \mu(u) + \text{constant} \tag{4}$$

where the constant ensures the condition $x(0) = 0$. The space domain of the coordinate x corresponding to D_u will be denoted by D. In the special frame we have $\tilde{g}_{00}(x) = -\tilde{g}_{11}(x)$ and $\tilde{\mu}(x) = 1$ which means that the scalar product (3) becomes just the usual one. Moreover, it is convenient to denote $\tilde{g}_{00} = 1 + v$ since then the form of the line element in the special frame,

$$\mathrm{d}s^2 = [1 + v(x)](\mathrm{d}t^2 - \mathrm{d}x^2) \tag{5}$$

gives us directly the relativistic potential, $V_R = m^2 v$, of the static Klein–Gordon equation,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_R(x)\right] U_E(x) = (E^2 - m^2) U_E(x). \tag{6}$$

Note that in the non-relativistic limit $V_R/2m$ becomes just the usual potential of the corresponding Schrödinger equation.

The linear operators we need will be introduced by using the special frames where the advantage is that the Hermitian conjugation is just the usual one (since $\tilde{\mu} = 1$). These are the coordinate and momentum operators,

$$(\boldsymbol{X}U)(x) = xU(x) \qquad (\boldsymbol{P}U)(x) = \mathrm{i}\frac{\mathrm{d}U(x)}{\mathrm{d}x} \tag{7}$$

and the Klein–Gordon operator,

$$\boldsymbol{H}^2 = m^2\mathbf{1} + \boldsymbol{\Delta}[V_R] \tag{8}$$

where $\boldsymbol{H}$ is the Hamiltonian operator defined by $\boldsymbol{H}U_E = EU_E$ and

$$\boldsymbol{\Delta}[V] = \boldsymbol{P}^2 + V(\boldsymbol{X}). \tag{9}$$

The whole algebra of observables is that freely generated by the operators $\boldsymbol{X}$ and $\boldsymbol{P}$ as in the Schrödinger picture of non-relativistic one-dimensional quantum mechanics.

3. Relativistic oscillators

The geometric models of ROs we discuss here are simple systems of test particles freely moving on static backgrounds which are able to simulate oscillations. This means that there are local charts of coordinates (t, u) where an observer at $u = 0$ moving along the direction ∂_t observes an oscillatory geodesic motion. These charts, called *proper* natural frames, have line elements [4],

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{11}\,\mathrm{d}u^2 = \frac{1 + (1+\lambda)\omega^2 u^2}{1 + \lambda\omega^2 u^2}\,\mathrm{d}t^2 - \frac{1 + (1+\lambda)\omega^2 u^2}{(1 + \lambda\omega^2 u^2)^2}\,\mathrm{d}u^2 \tag{10}$$

depending on a real parameter λ. Thus one obtains a family of metrics which are conformal transformations either of the anti-de Sitter metric (as given in [1]) or of the de Sitter one. The anti-de Sitter metric with $\lambda = -1$ is also included in this family. A special case is that of $\lambda = 0$ when we say that the line element

$$\mathrm{d}s^2 = (1 + \omega^2 u^2)(\mathrm{d}t^2 - \mathrm{d}u^2) \tag{11}$$

defines the *normal* RO. In [4] it is shown that the quantum models with $\lambda \leqslant 0$ have countable energy spectra, while for $\lambda > 0$ the energy spectra are mixed, with a finite discrete sequence and a continuous part. All of these models will be presented here in the special frames (t, x) associated with the proper frames (t, u) defined above. The advantage is that in the special frames our RO appear either as PT or as RM relativistic systems [6].

3.1. Relativistic Pöschl–Teller models

Let us consider first the models with $\lambda < 0$ when the metrics are conformal transformations of the anti-de Sitter one. We denote $\lambda = -\epsilon^2$ and $\hat{\omega} = \epsilon\, \omega$ (with $\epsilon \geqslant 0$) and calculate the space coordinate of the special frame. According to equation (4) we have

$$x = \frac{1}{\hat{\omega}} \arcsin \hat{\omega} u \tag{12}$$

while from equation (10) we obtain the line element in this frame,

$$\mathrm{d}s^2 = \left(1 + \frac{1}{\epsilon^2} \tan^2 \hat{\omega} x\right)(\mathrm{d}t^2 - \mathrm{d}x^2) \tag{13}$$

where the space domain is $D = (-\pi/2\hat{\omega}, \pi/2\hat{\omega})$. We recall that here we consider the universal covering spacetimes of the globally hyperbolic ones such that $t \in (-\infty, \infty)$. The relativistic potential,

$$V_{PT}(k, x) = \frac{m^2}{\epsilon^2} \tan^2 \hat{\omega} x = \hat{\omega}^2 k(k-1) \tan^2 \hat{\omega} x \tag{14}$$

is of PT type depending on the new parameter

$$k = \sqrt{\frac{m^2}{\epsilon^2 \hat{\omega}^2} + \frac{1}{4}} + \frac{1}{2} \tag{15}$$

which concentrates all the other ones. In the following we use k instead of m, as the main parameter of the PT models that will be denoted from now by (k).

The Klein–Gordon equation (6) of the model (k) with the potential (14) can be written as

$$\left[-\frac{1}{\hat{\omega}^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{k(k-1)}{\cos^2 \hat{\omega} x}\right] U(x) = \nu^2 U(x) \tag{16}$$

where

$$\nu^2 = \frac{E^2}{\hat{\omega}^2} - \left(1 - \frac{1}{\epsilon^2}\right) \frac{m^2}{\hat{\omega}^2} = \frac{E^2}{\hat{\omega}^2} + (1 - \epsilon^2) k(k-1). \tag{17}$$

This has only square-integrable solutions,

$$U_{k,n}(x) = N_{k,n} \sin^{2s} \hat{\omega} x \cos^k \hat{\omega} x\, F\left(-n_s, n_s + k + 2s, 2s + \tfrac{1}{2}, \sin^2 \hat{\omega} x\right) \tag{18}$$

for all $n_s = 0, 1, 2, \ldots$ and $2s = 0, 1$ [4, 10]. These define the *regular* modes whose energy levels,

$$E_{k,n}^2 = \hat{\omega}^2[(k+n)^2 + (\epsilon^2 - 1)k(k-1)] \tag{19}$$

depend on the main quantum number, $n = 2n_s + 2s$, which takes even values if $s = 0$ and odd values for $s = \frac{1}{2}$. Thus it results that the energy spectrum is countable, having no continuous part. In particular, for the anti-de Sitter model with $\epsilon = 1$ we recover the well known result $E_{k,n} = \omega(k+n)$ [3]. The normalization constants, $N_{k,n}$, will be calculated in the next section by using the supersymmetry and shape invariance [8] of the PT potentials.

Our PT models are well defined for any $k \in [1, \infty)$ since the limit $k \to 1$ (when $m \to 0$) has a good physical meaning. The model with $k = 1$ is interesting since it describes a massless particle confined to the rectangular infinite well of width $\pi/\hat{\omega}$. This has the equidistant energy levels

$$E_{1,n} = \hat{\omega}(n+1) \tag{20}$$

corresponding to the normalized eigenfunctions

$$U_{1,n}(x) = \sqrt{\frac{2\hat{\omega}}{\pi}} \sin(n+1)\left(\frac{\pi}{2} - \hat{\omega}x\right) \qquad n = 0, 1, 2, \ldots. \tag{21}$$

Note that this is a pure relativistic model since its non-relativistic limit (in the sense of special relativity) does not make sense.

3.2. Relativistic Rosen–Morse models

For $\lambda > 0$ the metrics of RO are conformal transformations of the de Sitter metric. Now we change the significance of ϵ and put $\lambda = \epsilon^2$ and $\hat{\omega} = \epsilon\omega$. Furthermore, from equation (4) we find

$$x = \frac{1}{\hat{\omega}} \operatorname{arcsinh} \hat{\omega}u \tag{22}$$

and from equation (10) we obtain the line elements

$$\mathrm{d}s^2 = \left(1 + \frac{1}{\epsilon^2}\tanh^2 \hat{\omega}x\right)(\mathrm{d}t^2 - \mathrm{d}x^2) \tag{23}$$

in special frames where the space domain is $D = (-\infty, \infty)$. These metrics define relativistic RM models whose potentials,

$$V_{RM}(j, x) = \frac{m^2}{\epsilon^2}\tanh^2 \hat{\omega}x = \hat{\omega}^2 j(j+1)\tanh^2 \hat{\omega}x \tag{24}$$

depend on the parameter

$$j = \sqrt{\frac{m^2}{\epsilon^2\hat{\omega}^2} + \frac{1}{4}} - \frac{1}{2}. \tag{25}$$

As in the case of PT models, we consider that j is the main parameter of the RM models, denoted by (j).

Now the Klein–Gordon equation is

$$\left[\frac{1}{\hat{\omega}^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{j(j+1)}{\cosh^2 \hat{\omega}x}\right] U(x) = \hat{\nu}^2 U(x) \tag{26}$$

where

$$\hat{\nu}^2 = -\frac{E^2}{\hat{\omega}^2} + \left(1 + \frac{1}{\epsilon^2}\right)\frac{m^2}{\hat{\omega}^2} = -\frac{E^2}{\hat{\omega}^2} + (1+\epsilon^2) j(j+1). \tag{27}$$

As in the non-relativistic case, the relativistic RM models have mixed energy spectra with a finite discrete sequence and a continuous part [4]. The solutions,

$$U_{j,n}(x) = N_{j,n}\sinh^{2s} \hat{\omega}x \cosh^{-j} \hat{\omega}x\, F\left(-n_s, n_s - j + 2s, 2s + \tfrac{1}{2}, -\sinh^2 \hat{\omega}x\right) \tag{28}$$

remain square-integrable only if the main quantum number, $n = 2n_s + 2s$ $(2s = 0, 1)$, takes the values $n = 0, 1, \ldots, n_{\max} < j$. Thus it gives a finite discrete energy spectrum included in the domain $[m, m\sqrt{1 + 1/\epsilon^2})$ with the energy levels

$$E_{j,n}^2 = \hat{\omega}^2[-(n-j)^2 + (\epsilon^2 + 1) j(j+1)]. \tag{29}$$

The definitive form of the normalized energy eigenfunctions of this spectrum will be calculated in the next section by using the shape invariance of the RM potentials. The continuous energy spectrum covers the domain $[m\sqrt{1 + 1/\epsilon^2}, \infty)$.

A special case is that of $m \to 0$ (when $j \to 0$). Then the discrete spectrum disappears while the continuous one becomes $[0, \infty)$. In this model the massless test particle moves like in flat spacetime and, therefore, there is no non-relativistic limit.

3.3. The normal RO and the non-relativistic limit

Our family of metrics is continuous in $\lambda = 0$ [4]. This means that the limits for $\epsilon \to 0$ of the PT and RM models must coincide. Indeed, according to equations (12) and (22) we find that in this limit $x \to u$ while from equations (15) and (25) it results that for any model with $m \neq 0$ we have $k \to \infty$, $j \to \infty$ but

$$\lim_{\epsilon\to 0} \epsilon^2 k = \lim_{\epsilon\to 0} \epsilon^2 j = \frac{m}{\omega}. \tag{30}$$

Furthermore, we can verify that the finite discrete spectra of the models with $\lambda > 0$ become countable, while the continuous spectra disappear such that all the PT and RM models have the same limit which is just the normal RO (with $\lambda = 0$). The special frame of this model coincides with the proper one where the metric is defined by equation (11). The relativistic potential is $V_0(u) = \lim_{\epsilon\to 0} V_{PT}(x) = \lim_{\epsilon\to 0} V_{RM}(x) = m^2\omega^2u^2$ so that the Klein–Gordon equation,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}u^2} + m^2\omega^2u^2\right] U_n^{(0)}(u) = \left(E_n^{(0)2} - m^2\right) U_n^{(0)}(u) \tag{31}$$

gives the familiar energy eigenfunctions of the NRHO,

$$U_n^{(0)} = \left(\frac{m\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{n!2^n}} \mathrm{e}^{-m\omega^2u^2/2} H_n(\sqrt{m\omega}\, u) \tag{32}$$

(where H_n are the Hermite polynomials), but relativistic energy levels,

$$E_n^{(0)2} = m^2 + 2m\omega\left(n + \tfrac{1}{2}\right). \tag{33}$$

In the non-relativistic limit, defined as in special relativity (for $m/\omega \to \infty$ and very small values of $E - m$), the normal RO becomes the NRHO with the potential $V_0/2m$ and usual energy levels. The non-relativistic limit of the other models, with $m \neq 0$ and $\lambda \neq 0$, can be easily calculated if we observe that, according to equations (15) and (25), this is equivalent to the limit $\lambda \to 0$ and, in addition, $m \gg \omega$. Hereby it results that all the RO with $m \neq 0$ have the same non-relativistic limit like the normal RO of mass m, namely the usual NRHO. On the other hand, it is interesting that in this way we can show that the parameter λ, or the parameter ϵ related to it, does not have a non-relativistic equivalent, since all the terms involving λ vanish in this limit.

4. Supersymmetry and shape invariance

A relativistic supersymmetric quantum mechanics can be constructed in the same way as the non-relativistic one. The main problem here is to find the operator which should play the role of a Hamiltonian. We show that this is just the operator (9) with the relativistic potential translated with an appropriate constant.

4.1. Supersymmetry

Let us start with a $(1+1)$-dimensional relativistic model with the potential V_R giving a finite or countable energy spectrum. First, we denote the energy levels by $E_n^{(-)}$ and the corresponding energy eigenfunctions by $U_n^{(-)}$. Then equation (6) in the special frame can be written as

$$\boldsymbol{\Delta}[V_-]U_n^{(-)} = d_n^{(-)} U_n^{(-)} \qquad n = 0, 1, 2, \ldots \tag{34}$$

where

$$V_- = V_R - (E_0^{(-)2} - m^2) \tag{35}$$

and

$$d_n^{(-)} = E_n^{(-)2} - E_0^{(-)2}. \tag{36}$$

We have translated the spectrum of $\boldsymbol{\Delta}$ in such a manner as to accomplish the condition $d_0^{(-)} = 0$ we need to define the superpotential [8]

$$W(x) = -\frac{1}{U_0^{(-)}} \frac{\mathrm{d}U_0^{(-)}(x)}{\mathrm{d}x}. \tag{37}$$

Then we have $V_- = W^2 - W'$ (with the notation $' = \partial_x$) and the supersymmetric partner (superpartner) potential of V_- reads $V_+ = W^2 + W' = -V_- + 2W^2$. Furthermore, we introduce the operator

$$\boldsymbol{A} = -\mathrm{i}\boldsymbol{P} + W(\boldsymbol{X}) \tag{38}$$

which satisfies

$$[\boldsymbol{A}, \boldsymbol{A}^\dagger] = 2W'(\boldsymbol{X}) \tag{39}$$

and helps us to write

$$\boldsymbol{\Delta}[V_-] = \boldsymbol{A}^\dagger \boldsymbol{A} \qquad \boldsymbol{\Delta}[V_+] = \boldsymbol{A}\boldsymbol{A}^\dagger. \tag{40}$$

Now, as in the non-relativistic case [8], we can convince ourselves that the spectrum of the eigenvalue problem

$$\boldsymbol{\Delta}[V_+]U_n^{(+)} = d_n^{(+)} U_n^{(+)} \tag{41}$$

coincides with that of equation (34), apart from the eigenvalue $d_0^{(-)} = 0$. Thus we have $d_n^{(+)} = d_{n+1}^{(-)}$, $n = 0, 1, 2, \ldots$, while the normalized eigenfunctions of $\boldsymbol{\Delta}[V_-]$ and $\boldsymbol{\Delta}[V_+]$ satisfy

$$\boldsymbol{A}U_n^{(-)} = \eta\sqrt{d_n^{(-)}}U_{n-1}^{(+)} \qquad \boldsymbol{A}^\dagger U_{n-1}^{(+)} = \eta^*\sqrt{d_n^{(-)}}U_n^{(-)} \tag{42}$$

where η is an arbitrary phase factor.

Hence, we can say that the $(1+1)$ relativistic supersymmetric quantum mechanics has the same main features as the non-relativistic one. It remains for us to study the shape invariance of the superpartner relativistic potentials.

4.2. Shape invariance

Let us consider the PT model (k) and identify $U_n^{(-)} \equiv U_{k,n}$ and $E_n^{(-)} \equiv E_{k,n}$. Then the differences (36) are

$$d_n^{(-)} \equiv d_{k,n} = E_{k,n}^2 - E_{k,0}^2 = \hat{\omega}^2 n(n+2k) \tag{43}$$

and from equations (35) and (14) we obtain

$$V_-(k, x) = V_{PT}(k, x) + m^2 - E_{k,0}^2 = \hat{\omega}^2[k(k-1)\tan^2\hat{\omega}x - k]. \tag{44}$$

On the other hand, the normalized ground-state eigenfunction calculated from equation (18),

$$U_{k,0}(x) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{1/4} \left[\frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})}\right]^{1/2} \cos^k \hat{\omega}x \tag{45}$$

gives the superpotential $W(k,x) = \hat{\omega}k \tan \hat{\omega}x$ which allows us to find the superpartner of V_-,

$$V_+(k,x) = -V_-(k,x) + 2W(k,x)^2 = \hat{\omega}^2[k(k+1)\tan^2 \hat{\omega}x + k]. \tag{46}$$

Moreover, with this superpotential the operator (38) reads

$$\boldsymbol{A}_k = -\mathrm{i}\boldsymbol{P} + \hat{\omega}k \tan \hat{\omega}\boldsymbol{X} = -\cos^k \hat{\omega}\boldsymbol{X}\,(\mathrm{i}\boldsymbol{P}) \cos^{-k} \hat{\omega}\boldsymbol{X} \tag{47}$$

while from equation (39) we obtain

$$[\boldsymbol{A}_k, \boldsymbol{A}_k^\dagger] = 2k\hat{\omega}^2\mathbf{1} + \frac{1}{2k}(\boldsymbol{A}_k + \boldsymbol{A}_k^\dagger)^2. \tag{48}$$

Now we observe that the potentials $V_-(k)$ and $V_+(k)$ are shape invariant since

$$V_+(k,x) = V_-(k+1,x) + \hat{\omega}^2(2k+1). \tag{49}$$

Consequently, we can identify $U_n^{(+)} \equiv U_{k+1,n}$ which means that the normalized energy eigenfunctions satisfy

$$\boldsymbol{A}_k U_{k,n} = \sqrt{d_{k,n}} U_{k+1,n-1} \qquad \boldsymbol{A}_k^\dagger U_{k+1,n-1} = \sqrt{d_{k,n}} U_{k,n} \tag{50}$$

as results from equations (40) with $\eta = 1$. Thus we have related the energy eigenfunctions of the model (k) with those of its superpartner model, $(k+1)$. In general, we can write any normalized energy eigenfunction of the model (k) as

$$U_{k,n} = \frac{1}{\hat{\omega}^n \sqrt{n!}} \left[\frac{\Gamma(n+2k)}{\Gamma(2n+2k)}\right]^{1/2} \boldsymbol{A}_k^\dagger \boldsymbol{A}_{k+1}^\dagger \dots \boldsymbol{A}_{k+n-1}^\dagger U_{k+n,0}. \tag{51}$$

where $U_{k+n,0}$ is the normalized ground-state eigenfunction of the model $(k+n)$ given by equation (45).

For the relativistic RM models we use the same method starting with the model (j) and denoting $U_n^{(-)} \equiv U_{j,n}$ and $E_n^{(-)} \equiv E_{j,n}$. Then the differences (36) are

$$d_n^{(-)} \equiv d_{j,n} = E_{j,n}^2 - E_{j,0}^2 = \hat{\omega}^2 n(2j-n) \tag{52}$$

and, according to (24), we have

$$V_-(j,x) = V_{RM}(j,x) + m^2 - E_{j,0}^2 = \hat{\omega}^2[j(j+1)\tanh^2 \hat{\omega}x - j]. \tag{53}$$

From equation (28) we find the normalized ground-state eigenfunction

$$U_{j,0}(x) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{1/4} \left[\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j)}\right]^{1/2} \cosh^{-j} \hat{\omega}x \tag{54}$$

giving the superpotential $W(j,x) = \hat{\omega}j \tanh \hat{\omega}x$. Hereby we obtain

$$V_+(j,x) = -V_-(j,x) + 2W(j,x)^2 = \hat{\omega}^2[j(j-1)\tanh^2 \hat{\omega}x + j]. \tag{55}$$

Now the operator (38) reads

$$\boldsymbol{A}_j = -\mathrm{i}\boldsymbol{P} + \hat{\omega}j \tanh \hat{\omega}\boldsymbol{X} = -\cosh^{-j} \hat{\omega}\boldsymbol{X}\,(\mathrm{i}\boldsymbol{P}) \cosh^j \hat{\omega}\boldsymbol{X} \tag{56}$$

while equation (39) gives

$$[\boldsymbol{A}_j, \boldsymbol{A}_j^\dagger] = 2j\hat{\omega}^2\mathbf{1} - \frac{1}{2j}(\boldsymbol{A}_j + \boldsymbol{A}_j^\dagger)^2. \tag{57}$$

The potentials $V_-(j)$ and $V_+(j)$ are shape invariant since

$$V_+(j, x) = V_-(j-1, x) + \hat{\omega}^2(2j-1). \tag{58}$$

Consequently, as in the previous case, we find that the normalized energy eigenfunctions satisfy

$$\boldsymbol{A}_j U_{j,n} = \sqrt{d_{j,n}}U_{j-1,n-1} \qquad \boldsymbol{A}_j^\dagger U_{j-1,n-1} = \sqrt{d_{j,n}}U_{j,n} \tag{59}$$

if we take $\eta = 1$ in equations (40). Thus we have obtained the relation between the sets of energy eigenfunctions of the superpartner models (j) and $(j-1)$. Moreover, we can also express the normalized eigenfunctions as

$$U_{j,n} = \frac{1}{\hat{\omega}^n\sqrt{n!}}\left[\frac{\Gamma(2j-2n+1)}{\Gamma(2j-n+1)}\right]^{1/2} \boldsymbol{A}_j^\dagger\boldsymbol{A}_{j-1}^\dagger\ldots\boldsymbol{A}_{j-n+1}^\dagger U_{j-n,0} \tag{60}$$

where now $U_{j-n,0}$ is the normalized ground-state eigenfunction of the model $(j-n)$ given by equation (54).

4.3. The normalized energy eigenfunctions

The normalization of the energy eigenfunctions of the PT models may be easily done in the usual way, but for the RM models there are some technical difficulties. These can be avoided by using the previous results since equations (51) and (60) are nothing other than the operator form of the Rodrigues formulae of the normalized eigenfunctions (in our phase convention with $\eta = 1$). Therefore, it remains only to rewrite their expressions in the usual form.

For the PT models we replace the operator (47) in equation (51) which takes the form

$$U_{k,n}(x) = \frac{(-1)^n}{\hat{\omega}^n\sqrt{n!}}\left[\frac{\Gamma(n+2k)}{\Gamma(2n+2k)}\right]^{1/2} \cos^{-k}\hat{\omega}x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos\hat{\omega}x}\frac{\mathrm{d}}{\mathrm{d}x}\cdots$$
$$\cdots\frac{1}{\cos\hat{\omega}x}\frac{\mathrm{d}}{\mathrm{d}x}\cos^{k+n-1}\hat{\omega}x\,U_{k+n,0}(x). \tag{61}$$

Then, according to equations (12) and (45), we obtain the final Rodrigues formula of the normalized energy eigenfunctions of the PT models in proper frames,

$$U_{k,n}(u) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{1/4}\frac{(-1)^n}{\hat{\omega}^n\sqrt{n!}}\left[\frac{\Gamma(2k+n)\Gamma(k+n+1)}{\Gamma(2k+2n)\Gamma(k+n+\frac{1}{2})}\right]^{1/2}$$
$$\times(1-\hat{\omega}^2u^2)^{-\frac{k-1}{2}}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(1-\hat{\omega}^2u^2)^{k+n-\frac{1}{2}}. \tag{62}$$

In the same way we can derive the Rodrigues formula for the normalized energy eigenfunctions of the RM models. By using equations (56) and (60) we find the normalized energy eigenfunctions of the discrete spectrum in proper frames,

$$U_{j,n}(u) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{1/4}\frac{(-1)^n}{\hat{\omega}^n\sqrt{n!}}\left[\frac{\Gamma(2j-2n+1)\Gamma(j-n+\frac{1}{2})}{\Gamma(2j-n+1)\Gamma(j-n)}\right]^{1/2}$$
$$\times(1+\hat{\omega}^2u^2)^{\frac{j+1}{2}}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(1+\hat{\omega}^2u^2)^{-j+n-\frac{1}{2}}. \tag{63}$$

Of course, as was expected, this formula gives square-integrable functions only for $n \leqslant n_{\max}$. On the other hand, here the problem of 'normalization' of the generalized energy eigenfunctions of the continuous spectrum remains open since there are no efficient procedures for doing this as yet.

Hence we have obtained the definitive formulae of the normalized energy eigenfunctions of our RO corresponding to the discrete energy levels. These can be written now in terms of Jacobi or Gegenbauer polynomials [16] and even as associated Legendre functions [13, 17], but only when k and j are integer numbers. In the particular case of the PT model with $k = 1$ the trigonometric form (21) can be derived from equation (62) by using the properties of the Tchebyshev polynomials.

In section 3.3 we have seen that the normal RO has the same energy eigenfunctions as the NRHO. Now, by taking into account that for large arguments, z, we have $\Gamma(z+a)/\Gamma(z+b) \sim z^{(a-b)}$ and by using equations (30) we verify directly that

$$\lim_{\epsilon\to 0} U_{k,n}(u) = \lim_{\epsilon\to 0} U_{j,n}(u) = U_n^{(0)}(u). \tag{64}$$

Because of these properties we can say that the eigenfunctions (62) and (63) represent relativistic generalizations of the NRHO eigenfunctions, different from that given by the algebraic method [2].

5. Shift operators

Our models have only one principal quantum number such that in each model we must have a pair of shift operators, i.e. the raising and lowering operators of the energy basis. In general, the shift operators are different from those of the supersymmetry apart from the shift operators of the normal RO that are up to factors just those of the supersymmetry since this model is its own superpartner.

Let us start with this simplest case since here the energy eigenfunctions are similar to those of the NRHO. Consequently, we can take over the well known results from the non-relativistic theory defining the differential operators

$$(\boldsymbol{a}U)(u) = \frac{1}{\sqrt{2\,m\omega}}\left(\frac{\mathrm{d}}{\mathrm{d}u} + m\omega u\right) U(u) \tag{65}$$

$$(\boldsymbol{a}^\dagger U)(u) = \frac{1}{\sqrt{2\,m\omega}}\left(-\frac{\mathrm{d}}{\mathrm{d}u} + m\omega u\right) U(u). \tag{66}$$

of the Heisenberg–Weyl algebra. Obviously, they are the desired shift operators which obey $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \mathbf{1}$, giving us the operator of number of quanta $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ and

$$\boldsymbol{X} = \frac{1}{\sqrt{2m\omega}}(\boldsymbol{a}^\dagger + \boldsymbol{a}) \qquad \boldsymbol{P} = -\mathrm{i}\sqrt{\frac{m\omega}{2}}(\boldsymbol{a}^\dagger - \boldsymbol{a}). \tag{67}$$

Moreover, we find the natural limits

$$\lim_{\epsilon\to 0} \boldsymbol{A}_k = \lim_{\epsilon\to 0} \boldsymbol{A}_j = \sqrt{2m\omega}\,\boldsymbol{a}. \tag{68}$$

For the models with $\lambda \neq 0$ the shift operators differ from those of supersymmetry. They can be calculated directly by using the action of the supersymmetry operators and the form of the normalized energy eigenfunctions derived above. In the case of $\lambda = -\epsilon^2$, after a few manipulations, we find that the shift operators of the PT model (k) can be defined in the proper frame as

$$(\boldsymbol{A}_{k,(+)}U_{k,n})(u) = \frac{1}{\hat{\omega}\sqrt{2k}}\left[-(1-\hat{\omega}^2u^2)\frac{\mathrm{d}}{\mathrm{d}u} + \hat{\omega}^2u(k+n)\right]U_{k,n}(u) \tag{69}$$

$$(\boldsymbol{A}_{k,(-)}U_{k,n})(u) = \frac{1}{\hat{\omega}\sqrt{2k}}\left[(1-\hat{\omega}^2u^2)\frac{\mathrm{d}}{\mathrm{d}u} + \hat{\omega}^2u(k+n)\right]U_{k,n}(u). \tag{70}$$

Their shifting action is

$$\boldsymbol{A}_{k,(+)}U_{k,n} = C^{(+)}_{k,n}U_{k,n+1} \qquad \boldsymbol{A}_{k,(-)}U_{k,n} = C^{(-)}_{k,n}U_{k,n-1} \tag{71}$$

where

$$C^{(+)}_{k,n} = \frac{1}{\sqrt{2k}}\left[\frac{(2k+n)(k+n)}{k+n+1}\right]^{1/2}\sqrt{n+1} \tag{72}$$

$$C^{(-)}_{k,n} = \frac{1}{\sqrt{2k}}\left[\frac{(2k+n-1)(k+n)}{k+n-1}\right]^{1/2}\sqrt{n}. \tag{73}$$

If we rewrite the action of the operators (69) and (70) in the special frame (t, x) then we recover the result of [13]. Furthermore, we can verify the commutation relation

$$[\boldsymbol{A}_{k,(-)}, \boldsymbol{A}_{k,(+)}]U_{k,n} = \left(1+\frac{n}{k}\right)U_{k,n} \tag{74}$$

and the identity

$$2k\boldsymbol{A}_{k,(+)}\boldsymbol{A}_{k,(-)}U_{k,n} = n(2k+n-1)U_{k,n} \tag{75}$$

which is just the Klein–Gordon equation in operator form [10]. In the limit $\epsilon \to 0$ we have [9]

$$\lim_{\epsilon\to 0}\boldsymbol{A}_{k,(+)} = \boldsymbol{a}^\dagger \qquad \lim_{\epsilon\to 0}\boldsymbol{A}_{k,(-)} = \boldsymbol{a}. \tag{76}$$

With the same procedure we find the shift operators of the RM model (j) in the proper frame,

$$(\boldsymbol{A}_{j,(+)}U_{j,n})(u) = \frac{1}{\hat{\omega}\sqrt{2j}}\left[-(1+\hat{\omega}^2u^2)\frac{\mathrm{d}}{\mathrm{d}u} + \hat{\omega}^2u(j-n)\right]U_{j,n}(u) \tag{77}$$

$$(\boldsymbol{A}_{j,(-)}U_{j,n})(u) = \frac{1}{\hat{\omega}\sqrt{2j}}\left[(1+\hat{\omega}^2u^2)\frac{\mathrm{d}}{\mathrm{d}u} + \hat{\omega}^2u(j-n)\right]U_{j,n}(u) \tag{78}$$

which have the action

$$\boldsymbol{A}_{j,(+)}U_{j,n} = C^{(+)}_{j,n}U_{j,n+1} \qquad \boldsymbol{A}_{j,(-)}U_{j,n} = C^{(-)}_{j,n}U_{j,n-1} \tag{79}$$

where

$$C^{(+)}_{j,n} = \frac{1}{\sqrt{2j}}\left[\frac{(2j-n)(j-n)}{j-n-1}\right]^{1/2}\sqrt{n+1} \tag{80}$$

$$C^{(-)}_{j,n} = \frac{1}{\sqrt{2j}}\left[\frac{(2j-n+1)(j-n)}{j-n+1}\right]^{1/2}\sqrt{n}. \tag{81}$$

They obey the commutation rule

$$[\boldsymbol{A}_{j,(-)}, \boldsymbol{A}_{j,(+)}]U_{j,n} = \left(1-\frac{n}{j}\right)U_{j,n} \tag{82}$$

and give us the operator form of the Klein–Gordon equation for discrete levels,

$$2j\boldsymbol{A}_{j,(+)}\boldsymbol{A}_{j,(-)}U_{j,n} = n(2j - n + 1)U_{j,n}. \tag{83}$$

Finally, we find that

$$\lim_{\epsilon\to 0}\boldsymbol{A}_{j,(+)} = \boldsymbol{a}^{\dagger} \qquad \lim_{\epsilon\to 0}\boldsymbol{A}_{j,(-)} = \boldsymbol{a}. \tag{84}$$

We must specify that the shift operators of the models with $\lambda \neq 0$ have two important properties; namely, they are not pure differential operators and, in addition, the raising and lowering operators are not adjoint to each other, i.e. $\boldsymbol{A}_{k,(\pm)} \neq (\boldsymbol{A}_{k,(\mp)})^{\dagger}$ and similarly for the RM models.

6. Comments

In this paper we have studied the quantum modes of a family of (1 + 1) RO by using the methods of a supersymmetric relativistic quantum mechanics similar to the well known non-relativistic one. This was possible since the form of the Klein–Gordon equation in the special frames is very close to that of the Schrödinger equation, allowing us to introduce the relativistic potentials and to exploit their shape invariance.

However, our relativistic theory has some new interesting features due to the fact that the mass is involved in the formula of the energy levels and, at the same time, play the role of a coupling constant. For this reason there are some regularities leading to a very simple parametrization such that for any pair of superpartner models we have either $\Delta k = \pm 1$ or $\Delta j = \mp 1$. Thus k and j simulate the behaviour of quantum numbers even though they cannot be considered as eigenvalues of self-adjoint operators [9]. On the other hand, the models with superpartner potentials can be seen as having particles of different masses moving on the same background. The consequence is that the masses of the sets of superpartner PT or RM models appear as being quantized according to the formulae $m_k^2 = \epsilon^2\hat{\omega}^2 k(k-1)$ and $m_j^2 = \epsilon^2\hat{\omega}^2 j(j+1)$, respectively. These remarkable properties helped us to easily write down the Rodrigues formulae of the normalized energy eigenfunctions of the discrete spectra and to find the corresponding shift operators.

In conclusion we can say that our family of models brings together the main solvable problems with parity-symmetric potentials of the one-dimensional quantum mechanics, interpreted as relativistic oscillators in the sense that all of these models (apart from those with $k = 1$ and $j = 0$) lead to the NRHO in the non-relativistic limit.

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